Math 275D Lecture 2 Notes

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1 Independence Properties and Construction of Brownian Motion

1.1 Independence of sections of Brownian motion

Denote independence by $\perp$. We know that $(B(t_2) - B(t_1)) \perp (B(s_2) - B(s_2))$ if $[s_1, s_2] \cap [t_1, t_2] = \varnothing$. However, this does not directly imply that the random variables $B(x)$ for $x \in [t_1, t_2]$ and $B(y)$ for $y \in [s_1, s_2]$ are independent. To prove this, we recall a consequence of the π-λ lemma.

Lemma 1.1. *Suppose $\mathcal{T}_i = \sigma(\mathcal{A}_i)$, where $\mathcal{A}_i$ is a π-system for $i = 1, 2$. If $A_1 \perp A_2$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, then $\mathcal{T}_1 \perp \mathcal{T}_2$.*

Proposition 1.1. *Let $t_1 < t_2 < s_1 < s_2$, and let $f(a) = B(t_1+a) - B(t_1)$ for $a \in [0, t_2 - t_1]$ and $g(b) = B(s_1+b) - B(s_1)$ for $b \in [0, s_2 - s_1]$. These two random functions are independent of each other.*

Remark 1.1. This is stronger than the fixed coordinates being independent because we can say things like $\max_a f(a) \perp \max_b g(b)$.

An important consequence of this is that for any t_0, Brownian motion from 0 to t_0 is independent of what happens after t_0.

Proposition 1.2. *Let $a > 0$. Then $B(at) \sim \sqrt{a} N(0, t)$.*

1.2 Difficulty in construction of Brownian motion

How can we construct Brownian motion? Recall that constructing $X \sim U[0, 1]$ is difficult; we have to talk about σ-fields and Lebesgue measure. A main difficulty is that not all sets are measurable. So we need to find a decent collection of measurable sets of functions for Brownian motion.

If we want to construct random vectors (X, Y), then we have to have $\mathcal{F} = \sigma(\mathcal{F}_X \times \mathcal{F}_Y)$. If we have a random sequence $(X_1, X_2, \dots)$, we have the σ-field $\sigma(\bigcup \mathcal{F}_i)$, but we need to use the Kolmogorov extension theorem[1] to construct $\mathbb{P}$.

With the Poisson process, we only needed to look at jumping times to understand the whole process. So we only need a sequence $(T_1, T_2, T_3, \dots)$. So we do not run into the same problem there we have with Brownian motion.

One idea (which does not work): Define $B(t), t \in \mathbb{Q}$ using the Kolmogorov extension theorem and extend the values continuously. But it is difficult to show that $\lim_{s \in \mathbb{Q} \to s_0} B(s)$ exists. So the correct idea is that we only get $B(t)$ for $t \in \mathbb{Z}[\frac{1}{2}]$ first, where $\mathbb{Z}[\frac{1}{2}] = \{m/2^{-n} : m, n \in \mathbb{Z}\}$ is the set of **dyadic rational numbers**.

Step 1: Using the Kolmogorov extension theorem, we can create a random list $C(x)$ for $x \in \mathbb{Q}_2$ such that

- $C(0) = 0$,
- separate intervals are independent,
- $C(y) - C(x) \sim N(0, y - x)$ for $x, y \in \mathbb{Z}[\frac{1}{2}]$.

Theorem 1.1. *$C(x)$ is uniformly continuous.*

We will prove this next time. Using this, the next step is as follows.

Step 2: Let $\psi : UCF(\mathbb{Z}[\frac{1}{2}]) \to C[0, 1]$ send $C(x)$ to its unique continuous extension. Then let $\mathbb{P}_{BM} = \mathbb{P}_{CM} \circ \psi^{-1}$.

1.3 Gaussian random vectors

Before we construct Brownian motion, we need to understand a notion related to Gaussian random variables.

Definition 1.1. A **Gaussian random vector** is a random vector $X = (X_1, \dots, X_n)$ such that for all $y \in \mathbb{R}^n$, $X \cdot y$ is a Gaussian random variable.

The reason we care about this is that $(B(1), B(2), B(3), \dots)$ is a Gaussian random vector (i.e. its finite dimensional projections are Gaussian random vectors).

Proposition 1.3. *Let X be a Gaussian random vector with $\mathbb{E}[X] = 0$. If $\mathbb{E}[X_i X_j] = 0$, then $X_i \perp X_j$.*

This does not hold for general random vectors and will be important for us in our construction of Brownian motion.

[1] Kolmogorov created the foundations for probability theory at the young age of 33.